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Stiffness of spherical bonded rubber bush mountings

J.M. Horton [a], G.E. Tupholme [b,*]

[a] *Reflecting Roadstuds Ltd., Boothtown, Halifax HX3 6TR, UK*
[b] *School of Computing and Mathematics, University of Bradford, Richmond Road, Bradford BD7 1DP, UK*

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Abstract

Exact closed-form expressions are derived for the torsional stiffnesses of spherical rubber bush mountings in the two principal modes of angular deformation, based upon the classical theory of elasticity. Agreement is found, as limiting cases, with the known results for the torsional stiffness and shear stiffness of an elastomer pad of circular cross-section.

Keywords: Rubber bushes; Stiffness; Spherical

1. Introduction

Major companies design and manufacture a full range of rubber bush mountings. These are utilized widely, but are often especially developed for improving the suspension characteristics of railway rolling stock and automobiles. Amongst these, spherical bushes play an important though specialized role, particularly in flexible couplings and floating ring couplings, where the accommodation of multidirectional movement is essential.

A spherical bush mounting consists of a shell of rubber which is bonded onto inner and outer rigid, concentric metallic spherical housings with their polar caps removed. The aim of this paper is to provide easily-calculable, closed-form expressions for their stiffness, and the stresses created, to aid their design, which at present is undertaken using "proven experience" according to manufacturers advertizing literature. The analyses presented here study the two principal torsional modes of deformation, in which torsional moments are applied to the outer surface with the inner surface maintained fixed. The only previous analysis of the situation in which the applied moment is about the z-axis was that undertaken by Hill (1975), but

* Corresponding author. Tel.: +44 1274 234273; fax: +44 1274 234282.
E-mail address: g.e.tupholme@bradford.ac.uk (G.E. Tupholme).

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unfortunately the expression he gives for the torsional stiffness appears to be fundamentally incorrect. There is apparently no corresponding analysis available for when the applied moment is about the x-axis.

The geometry of the bush and the fundamental governing equations are first formulated in Section 2. Then, in Section 3, exact analytical solutions are derived for the displacement and stress components created under a torsional deformation about the z-axis. A representation is determined for the corresponding torsional stiffness of the spherical bush, which is demonstrated in a limiting case to reproduce the torsional stiffness of an elastomer pad of circular cross-section. An analogous analysis is undertaken in Section 4 when the bush is subjected to a moment about the x-axis. An expression is found for the appropriate torsional stiffness, which is shown to reduce to the shear stiffness of a circular pad as a limiting case.

2. Physical formulation and governing equations

Consider a spherical rubber bush which is bonded to rigid concentric spherical metal surfaces at inner and outer radii a and b, respectively, and is bounded by the conical free surfaces $\theta = \alpha$ and $\theta = \pi - \alpha$, as shown in Fig. 1. Here a rectangular Cartesian coordinate system (x, y, z) is defined relative to an origin O at the centre of the bush, with the z-axis along the line $\theta = 0$, and is related to the spherical polar coordinates (r, θ, ϕ) of a point P within the bush by the equations

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta.$$

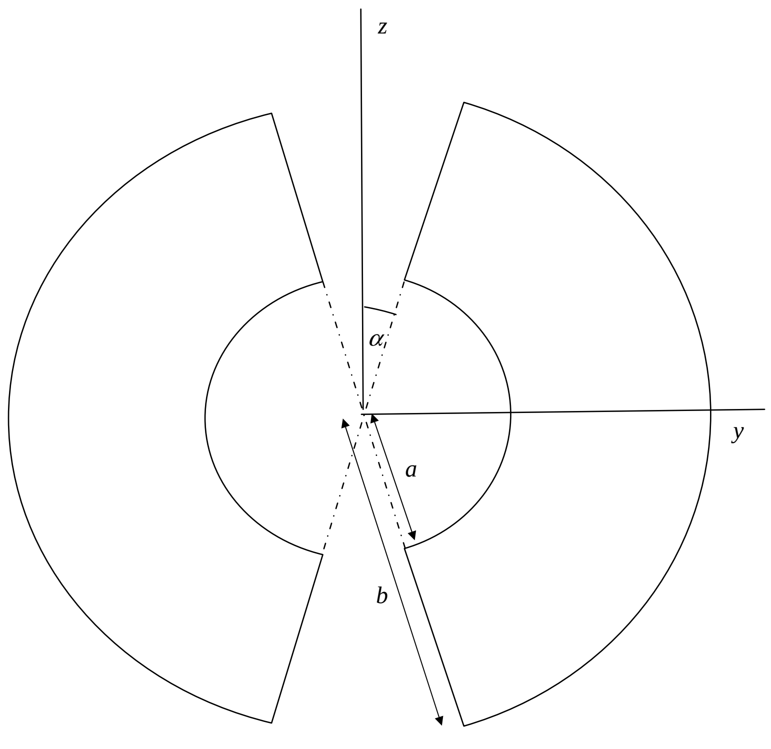

Fig. 1. Cross-section of the spherical bush through the $x = 0$ plane.

The inner surface is fixed throughout, while the outer surface is subjected to torsional moments in Sections 3 and 4. The objectives here are to derive expressions in each case for the subsequent components of the displacement and stress and the corresponding stiffnesses.

It is assumed that the rubber is homogeneous, isotropic and incompressible, and that the displacement gradients are sufficiently small during the deformations for the classical linear theory of elasticity to be applicable. Since the pioneering paper of Adkins and Gent (1954) in which they considered, both theoretically and experimentally, the behaviour of cylindrical rubber bush mountings, these assumptions have been widely regarded in the "rubber literature" as appropriate and justified in many practical applications. The radial, tangential and azimuthal components of the displacement of the point P are denoted by u_r, u_θ and u_ϕ, respectively, and the spherical strain and stress components by ε_{ij} and σ_{ij}, where $i, j = r$, θ or ϕ, in the usual notation.

For small strains, the assumption of incompressibility implies that

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi} = 0. \tag{1}$$

The strain–displacement gradient relations are

$$\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \\
&\varepsilon_{\phi\phi} = \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}, \\
&\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right), \\
&\varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right), \\
&\varepsilon_{\theta\phi} = \varepsilon_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r}\cot\theta + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta}\right),
\end{aligned} \tag{2}$$

and the constitutive equations can be written as

$$\begin{aligned}
&\varepsilon_{rr} = \frac{1}{3\mu}\left[\sigma_{rr} - \frac{1}{2}(\sigma_{\theta\theta} + \sigma_{\phi\phi})\right], \quad \varepsilon_{\theta\theta} = \frac{1}{3\mu}\left[\sigma_{\theta\theta} - \frac{1}{2}(\sigma_{rr} + \sigma_{\phi\phi})\right], \\
&\varepsilon_{\phi\phi} = \frac{1}{3\mu}\left[\sigma_{\phi\phi} - \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})\right], \quad \sigma_{r\theta} = \sigma_{\theta r} = 2\mu\varepsilon_{r\theta}, \\
&\sigma_{r\phi} = \sigma_{\phi r} = 2\mu\varepsilon_{r\phi}, \quad \sigma_{\theta\phi} = \sigma_{\phi\theta} = 2\mu\varepsilon_{\theta\phi},
\end{aligned} \tag{3}$$

where μ is the shear modulus. The equilibrium equations which must be fulfilled in the radial, tangential and azimuthal directions are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{2\sigma_{rr} + \sigma_{r\theta}\cot\theta - \sigma_{\theta\theta} - \sigma_{\phi\phi}}{r} = 0, \\
&\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta}{r} = 0, \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta}{r} = 0.
\end{aligned} \tag{4}$$

It is convenient as an aid to the following analyzes to note at this stage that elimination, using Eq. (1), between the expressions (3) leads to the relations

$$\sigma_{rr} = \sigma_{\theta\theta} + 2\mu(\varepsilon_{rr} - \varepsilon_{\theta\theta}),$$
$$\sigma_{\phi\phi} = \sigma_{\theta\theta} - 2\mu(\varepsilon_{rr} + 2\varepsilon_{\theta\theta}). \tag{5}$$

3. Torsional deformation about the z-axis

Suppose that, with the inner spherical surface fixed, a torsional moment of magnitude M^z is applied about the z-axis to the outer surface, thereby twisting it through a small angle Ω^z.

3.1. Solution for the torsional stiffness

It is clear that in this deformation mode the point P within the rubber will displace in the azimuthal direction along a circle of radius $r \sin\theta$ centred on, and at right angles to, the z-axis with the displacements in the radial and tangential directions both zero. Representations for the corresponding displacement components at the general point P in the rubber during this torsional deformation are thus sought in the forms

$$u_r = 0, \quad u_\theta = 0, \quad u_\phi = W(r)\sin\theta, \tag{6}$$

where the function W depends only upon r. The corresponding strain components follow using Eqs. (2). It is then seen that the incompressibility condition (1) is satisfied identically and, from Eqs. (5), that at any point

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi}, \tag{7}$$

and it is found from Eqs. $(3)_4$ to $(3)_6$ that the only non-zero shear stress component is given by

$$\sigma_{r\phi} = \mu\left(\frac{\mathrm{d}W}{\mathrm{d}r} - \frac{W}{r}\right)\sin\theta. \tag{8}$$

The equilibrium equations (4) thus yield the system

$$\frac{\partial\sigma_{\theta\theta}}{\partial r} = 0,$$
$$\frac{\partial\sigma_{\theta\theta}}{\partial\phi} + \mu\left(r\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + 2\frac{\mathrm{d}W}{\mathrm{d}r} - 2\frac{W}{r}\right)\sin^2\theta = 0, \tag{9}$$
$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0,$$

which is to be solved subject to the appropriate boundary conditions.

The general solutions of Eqs. (9) are

$$\sigma_{\theta\theta} = \alpha_1, \quad W = \alpha_2 r + \alpha_3/r^2, \tag{10}$$

where α_1, α_2 and α_3 are arbitrary constants. However, $\alpha_1 = 0$, since the bush has free surfaces at $\theta = \alpha$ and $\theta = \pi - \alpha$, and so in fact, from Eqs. (7),

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0 \tag{11}$$

everywhere within the rubber. Substitution of the representation $(10)_2$ into Eq. (8) gives

$$\sigma_{r\phi} = -\frac{3\mu\alpha_3}{r^3}\sin\theta. \tag{12}$$

The constant α_3 is determined by equating the combined moments about the z-axis of these components to the imposed couple M^z, through the expression

$$M^z = \int_0^{2\pi} \int_\alpha^{\pi-\alpha} \sigma_{r\phi} r^3 \sin^2\theta \, d\theta \, d\phi. \tag{13}$$

Evaluating this, using Eq. (12), shows that

$$\alpha_3 = -\frac{M^z}{12\mu\pi\cos\alpha\left(1 - \frac{1}{3}\cos^2\alpha\right)}. \tag{14}$$

The conditions

$$u_\phi = 0 \quad \text{for all } \theta \text{ when } r = a \tag{15}$$

and

$$u_\phi = b\sin\theta\Omega^z \quad \text{for all } \theta \text{ when } r = b \tag{16}$$

at the fixed inner surface and outer surface, respectively, imply by recalling Eqs. $(6)_3$ and $(10)_2$ that

$$\alpha_2 = -\frac{\alpha_3}{a^3} = \frac{b^3\Omega^z}{b^3 - a^3}. \tag{17}$$

A representation for the required torsional stiffness, T^z, defined by

$$T^z = \frac{M^z}{\Omega^z}, \tag{18}$$

can now be determined from Eqs. (14) and (17) in the form

$$T^z = \frac{12\mu\pi a^3 b^3}{b^3 - a^3}\cos\alpha\left(1 - \frac{1}{3}\cos^2\alpha\right). \tag{19}$$

It should be noted that this corrects the expression derived by Hill (1975, Eq. (4.7)), and subsequently re-quoted by Hill (1981, 1982) and Muhr (1995). Hill's error apparently arises through the omission of a multiplicative factor of $\sin\theta$ in the expression for w in his Eq. (4.1).

Putting $\alpha = 0$ into Eq. (19) gives the torsional stiffness, T_{Comp}, of a complete spherical shell as

$$T_{\text{Comp}} = \frac{8\mu\pi a^3 b^3}{b^3 - a^3}. \tag{20}$$

3.2. Stresses within the bush

The only non-zero stress component created in the bush is given by Eqs. (12) and (14) as

$$\sigma_{r\phi} = \frac{M^z \sin\theta}{4\pi r^3 \cos\alpha\left(1 - \frac{1}{3}\cos^2\alpha\right)}. \tag{21}$$

It attains its maximum value, $\sigma_{r\phi}^{\max}$, of

$$\sigma_{r\phi}^{\max} = \frac{M^z}{4\pi a^3 \cos\alpha\left(1 - \frac{1}{3}\cos^2\alpha\right)}, \tag{22}$$

when $r = a$ and $\theta = \pi/2$.

It is of especial interest to observe that the unbonded free surfaces $\theta = \alpha$ and $\theta = \pi - \alpha$ remain completely free of stress.

3.3. Limiting case

It is instructive to give added justification of the expression (19) for the torsional stiffness, T^z, by briefly demonstrating that it enables the known torsional stiffness of an elastomer pad of circular cross-section to be reassuringly reproduced as a limiting case.

The combined torsional stiffness of the two polar caps defined by $\theta \leqslant \alpha$ and $\theta \geqslant \pi - \alpha$ with $a \leqslant r \leqslant b$ for $0 \leqslant \phi \leqslant 2\pi$ is obtained by subtracting T^z, given by Eq. (19), from T_{Comp} in Eq. (20). Hence the torsional stiffness, T^z_{Pol}, of each polar cap is given by

$$T^z_{\mathrm{Pol}} = \frac{2\mu\pi a^3 b^3}{b^3 - a^3}(2 - 3\cos\alpha + \cos^3\alpha). \tag{23}$$

The cap for $\theta \leqslant \alpha$ is depicted in Fig. 2, with R as the arc distance from $\theta = 0$ to $\theta = \alpha$ along the mid-section of the region of thickness h.

It is clear from Fig. 2 that a, b, h, R and α are related by

$$a = \frac{R}{\alpha} - \frac{h}{2}, \quad b = \frac{R}{\alpha} + \frac{h}{2}. \tag{24}$$

As the radii a and b are increased, the shape of the region will tend towards that of a circular pad of radius R and thickness h. This process can be regarded as equivalent to taking the limit of T^z_{Pol} as the inclination, α, of the face $\theta = \alpha$ tends to zero. For small values of α, it follows from Eq. (23) that

$$T^z_{\mathrm{Pol}} \approx \frac{3\mu\pi a^3 b^3 \alpha^4}{2(b^3 - a^3)}. \tag{25}$$

Substitution of the relations (24) into the expression (25) leads to a power series expansion in α whose dominant leading term gives the limiting value

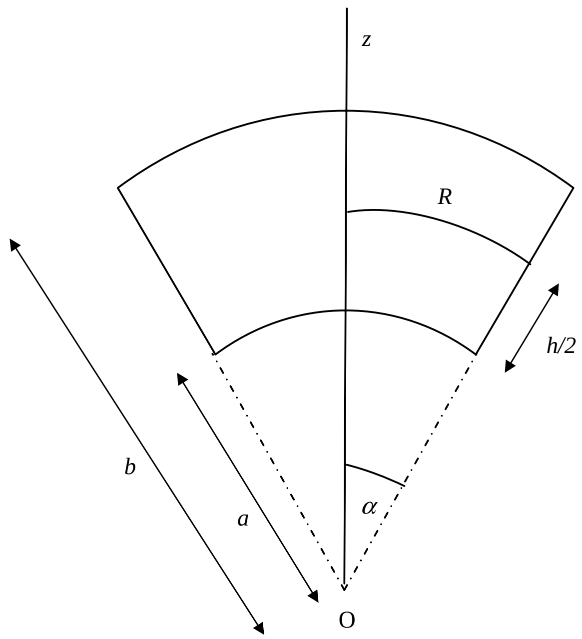

Fig. 2. Cross-section of the polar cap through the $x = 0$ plane.

$$\lim_{\alpha \to 0} T^z_{\text{Pol}} = \frac{\mu \pi R^4}{2h}. \tag{26}$$

This is recognized as the known torsional stiffness of a circular pad of radius R and thickness h as given by, for example, Muhr (1995).

4. Torsional deformation about the x-axis

Suppose now that a torsional moment of magnitude M^x is applied about the x-axis to the outer spherical surface causing it to twist through a small angle Ω^x, with the inner surface remaining fixed.

4.1. Solution for the torsional stiffness

For this mode of deformation the point P within the rubber will displace in the direction of Ω^x along a circle of radius $\sqrt{y^2 + z^2}$ centred on the x-axis and with zero displacement in the direction parallel to the x-axis. By resolving the polar displacements to form the Cartesian displacements it is found that in order for the deformation mode described to occur it is required that the solutions for the displacement components are now of the forms

$$u_r = 0, \quad u_\theta = V(r)\sin\phi, \quad u_\phi = V(r)\cos\theta\cos\phi, \tag{27}$$

with $V(r)$ being a function of r alone. These satisfy identically the incompressibility equation (1) and the corresponding stress components can be derived from Eqs. (5) and $(3)_4$ to $(3)_6$ as

$$\begin{aligned} &\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi}, \\ &\sigma_{r\theta} = \mu\left(\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{V}{r}\right)\sin\phi, \\ &\sigma_{r\phi} = \mu\left(\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{V}{r}\right)\cos\theta\cos\phi, \quad \sigma_{\theta\phi} = 0. \end{aligned} \tag{28}$$

The equilibrium equations (4) then reduce to the system

$$\begin{aligned} &\frac{\partial \sigma_{\theta\theta}}{\partial r} = 0, \\ &\frac{\partial \sigma_{\theta\theta}}{\partial \phi} + \mu\left(r\frac{\mathrm{d}^2V}{\mathrm{d}r^2} + 2\frac{\mathrm{d}V}{\mathrm{d}r} - 2\frac{V}{r}\right)\sin\theta\cos\theta\cos\phi = 0, \\ &\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \mu\left(r\frac{\mathrm{d}^2V}{\mathrm{d}r^2} + 2\frac{\mathrm{d}V}{\mathrm{d}r} - 2\frac{V}{r}\right)\sin\theta = 0. \end{aligned} \tag{29}$$

This has the general solutions

$$\sigma_{\theta\theta} = \beta_1, \quad V = \beta_2 r + \beta_3/r^2, \tag{30}$$

with β_1, β_2 and β_3 being arbitrary constants. But, since the bush has free surfaces at $\theta = \alpha$ and $\theta = \pi - \alpha$, $\beta_1 = 0$ which implies that

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0$$

at every point within the bush, and from Eqs. (28) the only non-zero shear stress components are

$$\sigma_{r\theta} = -\frac{3\mu\beta_3}{r^3}\sin\phi, \quad \sigma_{r\phi} = -\frac{3\mu\beta_3}{r^3}\cos\theta\cos\phi. \tag{31}$$

The value of the constant β_3 is derived by calculating the forces created in the bush by the stress components (31). It is found that the integrals between $\theta = \alpha$ and $\theta = \pi - \alpha$, with ϕ varying from 0 to 2π, of the components of the forces and their moments about the x-, y- and z-axis all vanish except for the moment about the x-axis which is given by

$$\int_0^{2\pi}\int_\alpha^{\pi-\alpha}[r\sin\theta\sin\phi(-\sin\theta\,\sigma_{r\theta}) - r\cos\theta(\cos\theta\sin\phi\,\sigma_{r\theta} + \cos\phi\,\sigma_{r\phi})]r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{32}$$

This must balance the applied moment, and hence evaluation of (32) using Eqs. (31) yields the relationship

$$\beta_3 = -\frac{M^x}{6\mu\pi\cos\alpha\left(1+\frac{1}{3}\cos^2\alpha\right)}. \tag{33}$$

The value of the remaining constant, β_2, is ascertained by imposition of the boundary conditions at the inner and outer surfaces. These can be expressed, respectively, in terms of the displacement components as

$$u_\theta = u_\phi = 0 \quad \text{for all } \theta \text{ and } \phi \quad \text{when } r = a \tag{34}$$

and

$$(u_\theta\cos\theta\sin\phi + u_\phi\cos\phi)\frac{b\cos\theta}{D^x} + (u_\theta\sin\theta)\frac{b\sin\theta\sin\phi}{D^x} = D^x\Omega^x$$
$$\text{for all } \theta \text{ and } \phi \text{ when } r = b, \tag{35}$$

where the distance, D^x, from a general point on the outer surface to the x-axis is given by

$$D^x = b\sqrt{\sin^2\theta\sin^2\phi + \cos^2\theta}. \tag{36}$$

Substitution of the expressions $(27)_2$ and $(27)_3$ combined with the solution $(30)_2$ into Eqs. (34) and (35) shows that

$$\beta_2 = -\frac{\beta_3}{a^3} = \frac{b^3\Omega^x}{b^3 - a^3}. \tag{37}$$

It follows from Eqs. (33) and (37) that the appropriate torsional stiffness, T^x, defined by

$$T^x = \frac{M^x}{\Omega^x}, \tag{38}$$

can be represented as

$$T^x = \frac{6\mu\pi a^3b^3}{b^3 - a^3}\cos\alpha\left(1 + \frac{1}{3}\cos^2\alpha\right). \tag{39}$$

As would be expected, the torsional stiffness, T_{Comp}, of a complete spherical shell is reproduced in the form (20) when the expression (39) is evaluated with $\alpha = 0$.

4.2. Stresses within the bush

Substitution of the value (33) for β_3 into Eqs. (31) shows that the only non-zero stress components produced in the bush are given by

$$\sigma_{r\theta} = \frac{M^x\sin\phi}{2\pi r^3\cos\alpha\left(1+\frac{1}{3}\cos^2\alpha\right)}, \tag{40}$$

$$\sigma_{r\phi} = \frac{M^x \cos\theta\cos\phi}{2\pi r^3 \cos\alpha\left(1+\frac{1}{3}\cos^2\alpha\right)}. \tag{41}$$

It is clear that the maximum magnitude, $\sigma_{r\phi}^{\max}$, of the stress component in Eq. (41) occurs on the inner spherical surface at $\theta = \alpha$ or $\pi - \alpha$ when $\phi = 0$ or π, where

$$\sigma_{r\phi}^{\max} = \frac{M^x}{2\pi a^3\left(1+\frac{1}{3}\cos^2\alpha\right)}, \tag{42}$$

while $\sigma_{r\theta}$ in Eq. (40) attains its maximum magnitude, $\sigma_{r\theta}^{\max}$, of

$$\sigma_{r\theta}^{\max} = \frac{M^x}{2\pi a^3 \cos\alpha\left(1+\frac{1}{3}\cos^2\alpha\right)} \tag{43}$$

at the inner surface when $\phi = \pi/2$ or $3\pi/2$. It should be noted however that, as observed in analogous analyses by other authors (see, for example, Gent (1992, p. 43)), in fact the stress component $\sigma_{r\theta}$ must actually physically decay rapidly to zero very near to the unbonded surfaces $\theta = \alpha$ and $\theta = \pi - \alpha$.

4.3. Limiting case

Utilizing analogous techniques to those of Section 3.3, the torsional stiffness T^x given by Eq. (39) can be shown to yield the shear stiffness of a circular pad. By calculating $T_{\mathrm{Comp}} - T^x$, using Eqs. (20) and (39), a measure of the shear stiffness, K_s^x, of the polar cap shown in Fig. 2 can be written as

$$K_s^x = \frac{F}{d} = \frac{\mu\pi a^3 b^3}{b^3 - a^3}(4 - 3\cos\alpha - \cos^3\alpha), \tag{44}$$

where d is the deflection of the polar cap when subjected to a shear force of magnitude F, so that

$$M^x = 2bF, \quad \Omega^x = d/b. \tag{45}$$

Taking small values of α in Eq. (44) and recalling the relations (24) leads to the limiting value

$$\lim_{\alpha\to 0} K_s^x = \frac{\mu\pi R^2}{h}. \tag{46}$$

This is the shear stiffness of a circular elastomer pad of radius R and thickness h.

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